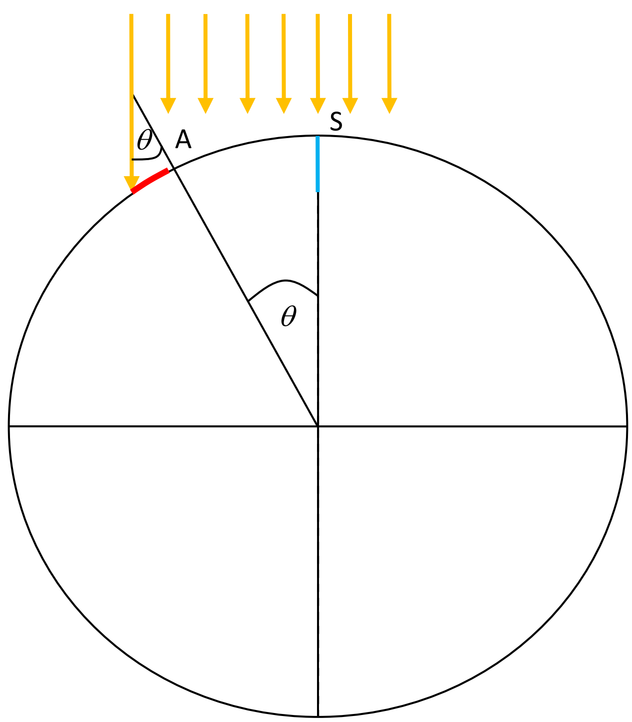
Teacher notes

Two truly spectacular measurements: the measurement of the radius of the earth by Eratosthenes and the measurement of the earth-moon distance by Aristarchus.

In about 240 BCE the Greek astronomer and mathematician Eratosthenes (276-194 BCE) measured the radius of the earth with extraordinary accuracy making only the simplest of measurements. This is one of the most ingenious measurements of all time. The Greeks knew that the earth was not flat from the simple observation of a ship disappearing over the horizon: the body of the ship disappears first and the mast last. The surface cannot be flat. Aristotle (384 – 322 BCE) noticed that in lunar eclipses the shape of the earth's shadow on the moon was **always** a circular arc. The only earth shape that can do this is a sphere. A flat disc-like earth would produce ellipses. Aristotle also mentions that when he had travelled to Egypt, he observed constellations that were not visible from Greece. That would not be the case if the earth was flat. But no one knew the radius of the earth until Eratosthenes.

The greatly exaggerated diagram shows a tower at A (Alexandria) and a well at S (Syene, modern day Aswan) two cities along the same meridian. On the summer solstice (June 21) the rays of the Sun are directly above S, and one can see the reflection of the Sun in the water in the well. (This happens because Syene is almost exactly on the Tropic of Cancer.) *At the same time*, at A, rays of the Sun cast a shadow of a tower. By measuring the length of the shadow (red arc) and the height of the tower the angle θ may be determined. The distance between Alexandria and Syene was known (measured by professionals called βηματιστές). The rays are parallel because the Sun is very far away.



From geometry, AS (the known Alexandria-Syene distance) is given by $AS = R\theta$ where θ is in radians or $AS = 2\pi R\frac{\theta}{360^\circ}$ if θ is in degrees. The ratio of the length of the shadow to the height of the pole was $\frac{1}{8}$ implying $\tan\theta = \frac{1}{8} \Rightarrow \theta = \arctan\frac{1}{8} = 7.1^\circ$. The known distance AS was 5000 stadia and so the circumference of the earth is

$$2\pi R = AS \times \frac{360^\circ}{7.1^\circ} \approx 250000 \text{ stadia.}$$

(Note that an angle of 7.1° is $\frac{7.1^\circ}{360^\circ} \approx \frac{1}{50}$ of a circle.)

A lot of records show that one stadion (stadium) was about 157 m making the circumference 39 250 km and a radius of 6250 km! The present value of the earth radius is 6370 km.

The calculation above is a modern calculation. This is not what Eratosthenes did.

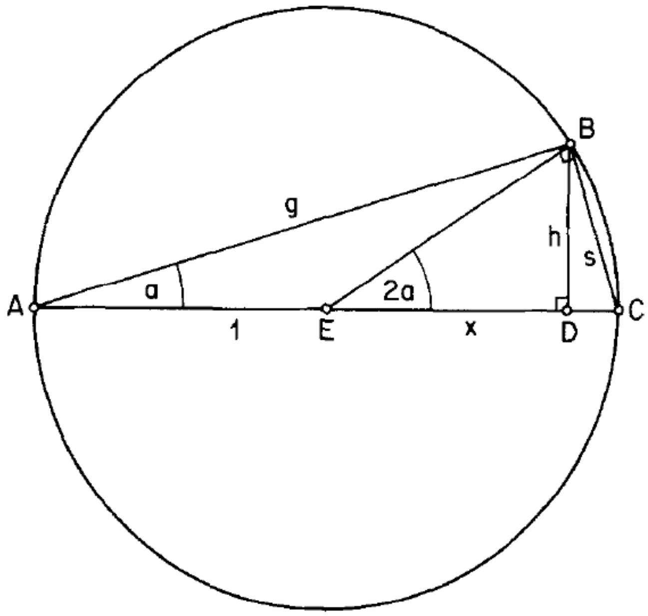
The first point is that the measurement of the shadow in Alexandria had to be made *at the same time* as the observation of the reflection of light in the well in Syene, namely when the Sun was directly overhead Syene. Eratosthenes had read on a papyrus in the library of Alexandria, where he was the director, that on June 21 (the summer solstice) a shadow would appear in Alexandria but not in Syene! Even if that information was not available, on the summer solstice the measurement of the shadow in Alexandria should have been made when the shadow was of least length. (Alexandria and Syene were on the same meridian and so the length of the shadow would be long in the morning decreasing to a minimum at noon and increasing again.)

The second point is the determination of the angle θ or, better, the ratio $\frac{1}{50}$ of a circle given that Eratosthenes did not know about the arctangent and he did not have a calculator!

The details that follow are for the interested reader and are not essential.

Using procedures which were well known to Greek mathematicians mainly due to the work of Archimedes and Aristarchus, Eratosthenes was able to deduce that the angle of 7.1° corresponded to about 1/50 of a full circle. The details follow for the interested reader.

Consider the following diagram taken from an article by B. Goldstein, in Historia Mathematica, **11** (1984), 411-416.



From triangles ABC, ABD and BDC we know that $\tan a = \frac{s}{g} = \frac{h}{1+x} = \frac{1-x}{h}$. Hence $h^2 = 1 - x^2$. Letting $2a = 30^\circ$ we know that $\frac{h}{x} = \frac{1}{\sqrt{3}}$. Hence, $h^2 = 1 - 3h^2 \Rightarrow h = \frac{1}{2}$. Then

$$\tan 15^\circ = \frac{h}{1+x} = \frac{\frac{1}{2}}{1 + \frac{1}{2} \times \sqrt{3}} = \frac{1}{2 + \sqrt{3}}.$$

Now, $\sqrt{3} = \sqrt{\frac{27}{9}} = \sqrt{\frac{25+2}{9}} = \frac{\sqrt{5^2+2}}{3} \approx \frac{5 + \frac{2}{2\times 5}}{3} = \frac{27}{15}$, where we used the fact that $\sqrt{a^2+b} = a\sqrt{1+\frac{b}{a^2}} \approx a(1+\frac{b}{2a^2}) = a + \frac{b}{2a}$. (The result $\sqrt{3} \approx \frac{27}{15}$ is only *approximate*. Root 3 being irrational cannot be expressed as a ratio of integers.)

Hence,

$$\tan 15^\circ \approx \frac{1}{2 + \frac{26}{15}} = \frac{15}{56} (\ = 0.2679)$$. A calculator gives $\tan 15^\circ = 0.2679$ (!) .

Repeating this procedure now for $2a = 15^\circ$ we find (the details are left as an exercise) $\tan 7.5^\circ \approx \frac{199}{1512} \approx \frac{5}{38}$. This angle ($7.5^\circ$) corresponds to 1/48 of a full circle. Thus, $\frac{s}{g} \approx \frac{5}{38}$. We need $\frac{s}{g} \approx \frac{1}{8}$.

By linear interpolation,

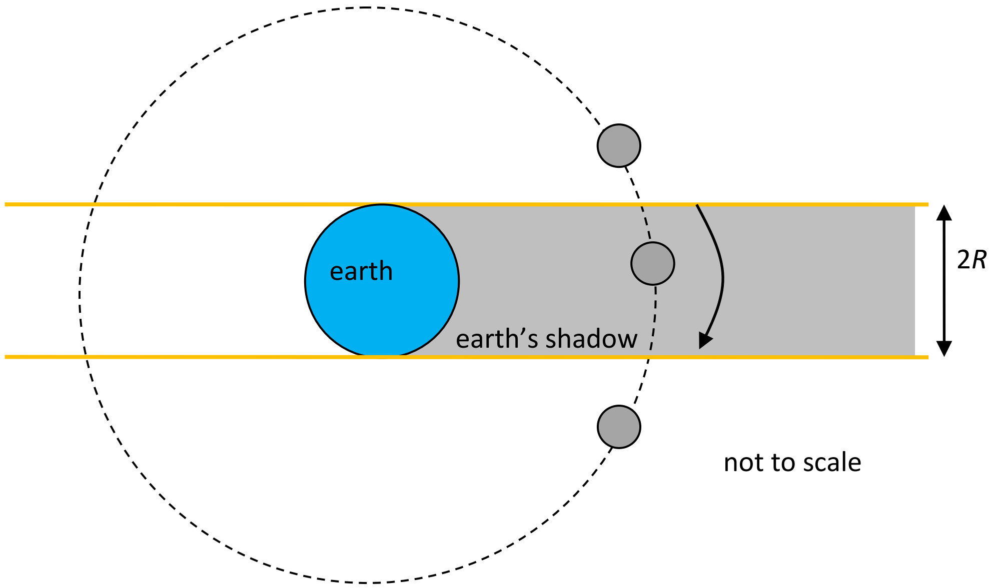
$$\frac{\frac{1}{8}}{\frac{5}{38}} = \frac{y}{\frac{1}{48}} \Rightarrow y \approx \frac{1}{50}.$$

In other words, the angle between the rays and the tower corresponds to an angle that is about $\frac{1}{50}$ of a circle and so the required circumference is 50×5000 or 250 000 stadia. This is how you circumvent the problem of the arctangent!

We now come to the moon. Aristotle had an argument like that used to show that the earth is round that shows that the moon is also round. The argument rests on the observation of lunar phases. The lunar phases are caused by the fact that the Sun illuminates part of the lunar surface. The shape of the curve separating the lit from the dark part of the surface (the terminator) is an ellipse. Only a spherical surface would cause elliptical terminators. For more details see the beautiful lecture by Terence Tao on https://www.youtube.com/watch?v=7ne0GArfeMs.

And now, to the second measurement, the earth-moon distance by Aristarchus (310-230 BCE), the man who was the first to postulate, some 1700 years before Copernicus, that the Sun and not the earth was the center of the solar system.

The diagram is an idealized diagram of the earth and the moon's orbit around the earth. The Sun is assumed to be very far away to the left of earth so that the earth's shadow has no penumbra. The width of the shadow is then $2R$ where R is the radius of the earth.



It was known to the Greeks that the longest duration of a lunar eclipse was 3.2 hours. The moon then covers a distance of approximately 2R in 3.2 hours and so its speed is $\frac{2R}{3.2}$. But the moon also completes one revolution around the earth in 27.3 days and so the speed of the moon is also $\frac{2\pi d}{27.3\times 24}$ where d is the earth-moon distance. Equating the two speeds gives

$$\frac{2\pi d}{27.3\times 24}=\frac{2R}{3.2}\Rightarrow d=\frac{27.3\times 24}{3.2\times \pi}R\approx 65R$$

The modern value of the average earth-moon distance is about 60R, so this is another impressive result.